

A note on some inequalities used in channel polarization and polar coding

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Abstract

We give a unified treatment of some inequalities that are used in the proofs of channel polarization theorems involving a binary-input discrete memoryless channel.

Let W be a binary-input discrete memoryless channel with $W(y|x)$ denoting the transition probability that output letter $y \in \mathcal{Y}$ is received given that input $x \in \{0, 1\}$ is sent. Assume without loss of generality that the channel is non-degenerate, i.e., $W(y|0) + W(y|1) > 0$ for every $y \in \mathcal{Y}$. Let the symmetric capacity be defined as:¹

$$I(W) := \sum_y \sum_{x \in \{0,1\}} \frac{1}{2} W(y|x) \log \frac{W(y|x)}{\frac{1}{2} W(y|0) + \frac{1}{2} W(y|1)}$$

and the Bhattacharyya parameter as:

$$Z(W) := \sum_y \sqrt{W(y|0)W(y|1)}$$

Below, we prove various inequalities relating the Bhattacharyya parameter to the symmetric capacity.

Let $\mathcal{H}(q) := -q \log(q) - (1-q) \log(1-q)$ denote the binary entropy function. Also define the Bhattacharyya function $\mathcal{B}(q) := 2\sqrt{q(1-q)}$. Both $\mathcal{H}(q)$ and $\mathcal{B}(q)$ are concave functions whose common domain and range are both equal to the interval $[0, 1]$. Define:

$$\phi : u \in [0, 1] \mapsto \mathcal{H}\left(\frac{1-\mathcal{B}(\frac{1-u}{2})}{2}\right) = \mathcal{H}\left(\frac{1-\sqrt{1-u^2}}{2}\right)$$

It can be verified that ϕ is a bijection and that $\phi(\mathcal{B}(q)) = \mathcal{H}(q)$ for all $q \in [0, 1]$. Anantharam *et al.* [AGKN13] studied ϕ in a different setting and showed that it is convex. We reprove this below and demonstrate other properties of ϕ that yield useful relationships between $I(W)$ and $Z(W)$ in a unified manner.

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¹log denotes the binary logarithm and ln denotes the natural logarithm.

Lemma 1. $0 < \phi''(u) < \phi'(u)/u$, for all $u \in (0, 1)$.

Proof. Let $v = \sqrt{1-u^2} \in (0, 1)$ to simplify the calculations. Taking derivatives of ϕ we obtain:

$$\frac{1}{u} \cdot \frac{d\phi}{du} = \frac{1}{\ln 2} \cdot \frac{\alpha(v)}{v} \quad (1)$$

$$\frac{d^2\phi}{du^2} = \frac{1}{\ln 2} \cdot \frac{\alpha(v) - v}{v^3}, \quad (2)$$

where $\alpha(v)$ above denotes the inverse hyperbolic tangent function, i.e., $\alpha : v \in (0, 1) \mapsto \frac{1}{2} \log\left(\frac{1+v}{1-v}\right)$.

The Taylor series of $\alpha(v)$ equals $\sum_{n \geq 1} \frac{v^{2n-1}}{2n-1}$ which converges absolutely for $v \in (0, 1)$. Therefore:

$$\begin{aligned} \frac{\phi'(u)}{u} &= \frac{1}{\ln 2} \cdot \left(1 + \sum_{n \geq 1} \frac{v^{2n}}{2n+1}\right) \\ \phi''(u) &= \frac{1}{\ln 2} \cdot \left(\frac{1}{3} + \sum_{n \geq 1} \frac{v^{2n}}{2n+3}\right) \end{aligned}$$

Comparing the right hand side of both expressions term by term, the desired inequality follows for all $u \in (0, 1)$. \square

Lemma 2. *The function $\phi(u)$ is strictly convex whereas the function $\phi(\sqrt{w})$ is strictly concave over their domain $[0, 1]$.*

Proof. Since $\phi(u)$ is continuous over its domain $[0, 1]$, and $\phi''(u) > 0$ for all $u \in (0, 1)$ by Lemma 1, thus $\phi(u)$ is strictly convex.

Define $\psi(w) := \phi(\sqrt{w})$ and let $u = \sqrt{w}$. Now $\psi''(w) = \frac{1}{4u^2} \cdot (\phi''(u) - \phi'(u)/u) < 0$ by Lemma 1, for all $u \in (0, 1)$. Since $\psi(w)$ is also continuous over $[0, 1]$, it is strictly concave. \square

As a consequence, we obtain the following inequalities.

Lemma 3. *For all $u \in [0, 1]$:*

- (a) $\phi(u) \leq u$ with equality only at $u \in \{0, 1\}$;
- (b) $\phi(u) \geq u^2$ with equality only at $u \in \{0, 1\}$; and
- (c) $\phi(u) \geq 1 + (u - 1)/\ln 2$ with equality only at $u = 1$.

Lemma 3(a) can be restated as $\mathcal{H}(q) \leq \mathcal{B}(q)$, as shown by Lin [Lin91, Theorem 8]. Lemma 3(b) can be restated as $\mathcal{H}(q) \geq \mathcal{B}(q)^2$, as shown by Arikan [Ari10]. The lower bounds given in Lemma 3(b) and Lemma 3(c) are incomparable: when $u = 0$, Lemma 3(b) is tight but not Lemma 3(c); when $u = 1 - \varepsilon$ for some small $\varepsilon > 0$, then $\phi(u) = 1 - \varepsilon \log e + \Theta(\varepsilon^2)$. Up to the linear term this matches the bound given by Lemma 3(c) but we get a worse bound with Lemma 3(b).

Proof (of Lemma 3). The proof uses the convexity statements in Lemma 2. The inequality in part (a) follows by convexity: $\phi(u) \leq (1-u) \cdot \phi(0) + u \cdot \phi(1) = u$. Note that $\phi(u) - u = 0$ for $u \in \{0, 1\}$ and by strict convexity of the function $\phi(u) - u$, this value is achieved only at the end points.

The inequality in part (b) follows by concavity: $\phi(\sqrt{w}) \geq (1-w) \cdot \phi(\sqrt{0}) + w \cdot \phi(\sqrt{1}) = w$; now set $w = u^2$. By strict concavity, the minimum of $\phi(\sqrt{w}) - w$ is achieved only at the end points so equality holds only at $w = u \in \{0, 1\}$.

For part (c), let $\ell(u)$ denote the right side of the inequality. We show that $\ell(u)$ is the tangent line at $u = 1$ which by convexity would establish the inequality. By definition the tangent at $u = 1$ equals $\phi(1) + (u-1)\phi'(1)$ so we need to show that $\phi'(1) = \frac{1}{\ln 2}$. By eq. (1), we have:

$$\begin{aligned} \phi'(1) &= \lim_{u \rightarrow 1} \frac{\phi'(u)}{u} = \lim_{x \rightarrow 0} \frac{\alpha(x)}{x \ln 2} \\ &= \frac{1}{\ln 2} \cdot \lim_{x \rightarrow 0} \alpha'(x) = \frac{1}{\ln 2} \cdot \lim_{x \rightarrow 0} \frac{1}{1-x^2} = \frac{1}{\ln 2} \end{aligned}$$

Now $\phi(u) = \ell(u)$ at $u = 1$ and by strict convexity of $\phi(u) - \ell(u)$, its minimum is achieved only at this point. \square

The above properties of ϕ have the following implications for relating $I(W)$ to $Z(W)$. Under the uniform distribution on the input $\{0, 1\}$, let Y denote the output induced by the channel, i.e., for each output letter $y \in \mathcal{Y}$, $p_Y(y) = \frac{1}{2}(W(y|0) + W(y|1))$. Define the random variable:

$$U(y) := \mathcal{B}(Q(y)), \quad \text{where } Q(y) := \frac{W(y|0)}{W(y|0) + W(y|1)}$$

The law of Q is referred to as the Blackwell measure of W in [Rag16]. Related measures, giving alternative characterizations of a binary-input memoryless channel, have been used extensively in the context of information combining in [TR08, Ch. 4], and more specifically in polar coding in [Saş12, p. 30].

Rewrite the channel parameters $I(W)$ and $Z(W)$ as expectations of appropriate functions of U :

$$\begin{aligned} Z(W) &= \sum_y p_Y(y) \mathcal{B}(Q(y)) = \mathbb{E} \mathcal{B}(Q) = \mathbb{E} U \\ 1 - I(W) &= \sum_y p_Y(y) \mathcal{H}(Q(y)) = \mathbb{E} \mathcal{H}(Q) = \mathbb{E} \phi(U) \end{aligned} \tag{3}$$

Theorem 4. $Z(W) \geq 1 - I(W) \geq \phi(Z(W))$

Proof. Applying Lemma 3(a) and then using the fact that ϕ is convex (Lemma 2) yields: $\mathbb{E} U \geq \mathbb{E} \phi(U) \geq \phi(\mathbb{E} U)$. Now substitute the identities in eq. (3). \square

By Lemma 3, the first inequality is tight iff $U \in \{0, 1\}$ with probability 1. In other words, the inequality is tight iff the channel W is such that $W(y|0)W(y|1) = 0$ or $W(y|0) = W(y|1)$

for each output y . A channel with this property is called a binary erasure channel (BEC). Indeed, this inequality was proved by Arikan [Ari09, Prop. 11] by an indirect argument, using an extremal property of the BEC in channel polarization.

The second inequality is tight iff U is constant with probability 1. Divide the outputs into two classes based on the predicate $W(y|0) > W(y|1)$; this is operationally equivalent to a binary symmetric channel (BSC), i.e., a binary-input channel for which there exists a constant $0 \leq \epsilon \leq \frac{1}{2}$ such that each y satisfies $\epsilon \cdot W(y|x) = (1 - \epsilon) \cdot W(y|1 - x)$ for some $x \in \{0, 1\}$.

Now Lemma 3(b) implies that $\phi(Z(W)) \geq Z(W)^2$ so we obtain: $1 - I(W) \geq Z(W)^2$ (cf. [Ari10]). Equality holds only when $Z(W) \in \{0, 1\}$. Equivalently, the distributions $W(\cdot|0)$ and $W(\cdot|1)$ are either identical or have disjoint support. Next Lemma 3(c) implies that $I(W) + Z(W) \cdot \log e \leq \log e$. Equality holds only when $Z(W) = 1$, i.e., the distributions $W(\cdot|0)$ and $W(\cdot|1)$ are identical. To summarize:

Corollary 5. *For a binary input symmetric channel W :*

- (a) $I(W) + Z(W) \geq 1$. *Equality holds only for the BEC.*
- (b) $I(W) + \phi(Z(W)) \leq 1$. *Equality holds only for the BSC.*
- (c) $I(W) + Z(W)^2 \leq 1$. *Equality holds iff $Z(W) \in \{0, 1\}$.*
- (d) $I(W) \cdot \ln 2 + Z(W) \leq 1$. *Equality holds iff $Z(W) = 1$.*

Finally, we note that these inequalities can be restated in terms of distances between probability distributions, which was the original motivation of Lin [Lin91]. Let P and Q be two distributions P and Q on \mathcal{Y} . Identify $W(\cdot|0)$ with P and $W(\cdot|1)$ with Q . Then the Hellinger distance $H(P, Q)$ equals $\sqrt{1 - Z(W)}$ and the Jensen–Shannon divergence $JS(P, Q)$ equals $I(W)$. Thus Corollary 5 can be restated as follows:

Proposition 6. *For two distributions P and Q :*

$$H^2(P, Q) \leq JS(P, Q) \leq H^2(P, Q) \cdot \min(\log e, 2 - H^2(P, Q))$$

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